Singular statistics

Eugène Bogomolny,[1] Ulrich Gerland,[2] and Charles Schmit[1]
[1]*Laboratoire de Physique Théorique et Modèles Statistiques, Unité de Recherche de l'Université Paris XI et du CNRS (UMR 8626), Université Paris-Sud, 91405 Orsay Cedex, France*
[2]*Physics Department, University of California San Diego, La Jolla, California 92093-0319*
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We consider the statistical distribution of zeros of random meromorphic functions whose poles are independent random variables. It is demonstrated that correlation functions of these zeros can be computed analytically, and explicit calculations are performed for the two-point correlation function. This problem naturally appears in, e.g., rank-1 perturbation of an integrable Hamiltonian and, in particular, when a δ-function potential is added to an integrable billiard.

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I. INTRODUCTION

The investigation of statistical properties of quantum energy levels of a given system is a long-standing problem (see, e.g., Refs. [1–3]). According to accepted conjectures energy levels of integrable systems behave as independent random variables (i.e., they obey Poisson statistics) [4], and those of generic chaotic systems follow the random matrix predictions [5]. The proof of these conjectures in the full generality is without doubt quite difficult, and is still lacking, though partial results (concerning mostly integrable models) are available (see, e.g., Ref. [6] and references therein). But there are systems which are neither integrable nor completely chaotic for which quantum energy levels are defined by an equation

$$f(E)=0, \tag{1}$$

with a well defined (and simple) function $f(E)$. In Ref. [7] the case of a polynomial equation

$$f(E)=\sum_{n=0}^{N} a_n E^n \tag{2}$$

was considered, and statistical properties of solution of $f(E)=0$ were calculated provided a_n be independent random variables.

The purpose of this paper is to consider the case of random meromorphic functions of the form

$$f(E)=P(E)+\sum_{j=1}^{N}\frac{r_j}{E-e_j}, \tag{3}$$

where $P(E)$ is a polynomial, and e_j and r_j are, correspondingly, poles and residues of $f(E)$.

The natural example leading to the quantization condition in this form is the perturbation of a Hamiltonian by a rank-1 perturbation. If $H^{(0)}_{\mu\nu}$ is an unperturbed Hamiltonian, then the Hamiltonian after perturbation is

$$H_{\mu\nu}=H^{(0)}_{\mu\nu}+v_\mu v_\nu, \tag{4}$$

where v_μ is a perturbation vector.

Solutions of the ''Schrödinger'' equation

$$H_{\mu\nu}\psi_\nu=E\psi_\mu \tag{5}$$

can be expressed through solutions of the unperturbed equation

$$H^{(0)}_{\mu\nu}\psi^{(0)}_\nu(n)=e_n\psi^{(0)}_\mu(n) \tag{6}$$

as

$$\psi_\mu=\sum_n c_n\psi^{(0)}_\mu(n), \tag{7}$$

where (up to a factor)

$$c_n=\frac{\langle v|\psi^{(0)}(n)\rangle}{E-e_n}, \tag{8}$$

provided new eigenvalues E obey the following quantization condition:

$$\sum_n \frac{|\langle v|\psi^{(0)}(n)\rangle|^2}{E-e_n}=1. \tag{9}$$

Here $\langle v|\psi^{(0)}(n)\rangle=\Sigma_\mu v_\mu\psi^{(0)}_\mu(n)$.

This equation has the form of Eq. (3) with $P(E)=\text{const}$, while unperturbed energy levels play the role of poles, and the residues are projections of unperturbed wave functions in the directions of the perturbation vector

$$r_n=|\langle v|\psi^{(0)}(n)\rangle|^2. \tag{10}$$

The addition of a δ-function potential

$$V(x)=\lambda\,\delta(\vec{x}-\vec{x}_0) \tag{11}$$

corresponds exactly to a rank-1 perturbation. In this case (see, e.g., Refs. [8,9]), Eq. (9) takes the form

$$\lambda\sum_n \frac{|\psi^{(0)}_n(\vec{x}_0)|^2}{E-e_n}=1, \tag{12}$$

where $\psi_n^{(0)}(\vec{x})$ and e_n are eigenfunctions and eigenvalues of the problem without the δ-function potential.

Another model which leads to similar equations is the Bohr-Mottelson model [10], which describes the interaction of one level (denoted below by index 0) with all other levels. The model is defined by the Hamiltonian

$$H=H_0+V, \tag{13}$$

where the interaction potential has nonzero matrix elements only between the chosen level and all other levels:

$$V_{0i}=V_{i0}, \quad V_{00}=V_{ij}=0. \tag{14}$$

The energy levels of Hamiltonian (13) obey the equation [10]

$$\sum_j \frac{|V_{0j}|^2}{E-e_j}-(E-e_0)=0, \tag{15}$$

which is again of the form of Eq. (3), with a linear polynomial part.

A quite natural question appears: What is the statistical distribution of the new eigenvalues [i.e., solutions of Eq. (3)] provided that statistical distributions of poles and residues are known? In Ref. [11] it was proved that, if the unperturbed system is described by random matrix theory, the distribution of new eigenvalues will also be of random matrix type.

The main purpose of this paper is to compute analytically the statistical distribution of solutions of Eq. (3) when the poles e_j are independent random variables (i.e., obey the Poisson statistics). We shall show that in this case the resulting statistics exhibits a level repulsion and differs from known distributions.

The plan of the paper is the following. In Sec. II the general formalism is described. In Sec. III a calculation of the mean density is presented. In Sec. IV the two-point correlation function is computed when all residues r_j in Eq. (3) are the same. Generalization to different residues is discussed in Sec. VII. As the exact expression of the two-point correlation function is cumbersome, in Sec. V the series expansion of the results is given. In Sec. VI the limiting behavior of the two-point correlation function for small and large energy differences is obtained without knowledge of the exact solution. The details of the calculation of a certain important integral are presented in the Appendix.

II. GENERAL FORMALISM

We consider the most interesting case of Eq. (3) when the mean separation of the poles is much smaller than a characteristic scale of polynomial $P(E)$. Under such a condition this polynomial can be considered as a constant, and after dividing by it Eq. (3) takes the form

$$\sum_{j=1}^{N} \frac{r_j}{E-e_j}=1. \tag{16}$$

Our goal is to find the statistical distribution of solutions E of this equation, provided r_j are constants and N numbers e_j are independent random variables with a common distribution $d\mu(e)$, which for simplicity we choose as follows:

$$d\mu(e)=\begin{cases} \dfrac{1}{2W}de & \text{if } -W\leqslant e_j\leqslant W \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

As the density of these poles is a constant, they can be considered as eigenvalues of a two-dimensional integrable billiard, and we shall call them energy levels (or unperturbed energy levels). All our calculations also remain valid in a more general case when the mean density of poles is not a constant but is not changed noticeably in the scale of the mean pole separation (e.g., for three-dimensional integrable models). The only difference is that $N/2W$ below should be substituted for by the local mean density of poles, $\overline{\rho}$ (see the end of Sec. III).

In general, if one is interested in solutions of the equation

$$f(x_n)=0, \tag{18}$$

it is often convenient to express the exact density of such solutions,

$$\rho(x)=\sum_n \delta(x-x_n), \tag{19}$$

in the following manner:

$$\rho(x)=\delta(f(x))\left|\frac{df(x)}{dx}\right|. \tag{20}$$

The main advantage of such a representation is the possibility of calculating the statistical distribution of roots x_n, directly from statistical distribution of coefficients of $f(x)$. This method has been used for deriving the distribution of roots of random polynomials [7].

In our case,

$$\rho(E)=\delta\left(\sum_{j=1}^{N}\frac{r_j}{E-e_j}-1\right)\sum_{k=1}^{N}\frac{r_j}{(E-e_j)^2}. \tag{21}$$

Representing the δ function as the Fourier integral (i.e., considering the characteristic function of the roots), one obtains

$$\rho(E)=\int_{-\infty}^{\infty}\frac{d\alpha}{2\pi}\exp\left[i\alpha\left(\sum_{j=1}^{N}\frac{r_j}{E-e_j}-1\right)\right]\sum_{k=1}^{N}\frac{r_k}{(E-e_k)^2}. \tag{22}$$

It is this representation of the exact density that we shall use throughout the paper. As all e_j's are considered as independent random variables, this expression can be rewritten in the form

$$\rho(E)=\int\frac{d\alpha}{2\pi}e^{-i\alpha}\prod_{j=1}^{N}\exp\left(i\frac{\alpha r_j}{E-e_j}\right)\sum_{k=1}^{N}\frac{r_k}{(E-e_k)^2}, \tag{23}$$

where all factors are also independent random variables which clearly permit one to find all mean values by straightforward integration.

III. MEAN DENSITY

Let us start with a calculation of the mean density

$$\langle \rho(E)\rangle = \int \frac{d\alpha}{2\pi} e^{-i\alpha} \prod_{j=1}^{N} \int d\mu(e_j) \exp\left(i \frac{\alpha r_j}{E-e_j} \right) \times \sum_{k=1}^{N} \frac{r_k}{(E-e_k)^2}. \tag{24}$$

The integrals can be transformed as

$$\langle \rho(E)\rangle = \int \frac{d\alpha}{2\pi} e^{-i\alpha} \left(\sum_{k=1}^{N} r_k g(r_k\alpha) \prod_{j\neq k} f(r_j\alpha) \right), \tag{25}$$

where

$$f(\alpha) = \int d\mu(e) \exp\left(i \frac{\alpha}{E-e} \right) \tag{26}$$

and

$$g(\alpha) = \int d\mu(e) \frac{1}{(E-e)^2} \exp\left(i \frac{\alpha}{E-e} \right) = -\frac{\partial^2}{\partial \alpha^2} f(\alpha). \tag{27}$$

Let us rewrite the expression for $f(\alpha)$ in the form

$$f(\alpha) = 1 - \frac{1}{2W} I(\alpha), \tag{28}$$

where

$$I(\alpha) = \int_{-W}^{W} de \left[1 - \exp\left(i \frac{\alpha}{E-e} \right) \right]. \tag{29}$$

As

$$g(\alpha) = \frac{1}{2W} \frac{\partial^2}{\partial \alpha^2} I(\alpha), \tag{30}$$

it is necessary to compute only $I(\alpha)$.

Though the above steps are exact for finite N, the most interesting case is the case $N\to\infty$. In this limit only small values of α are important ($\alpha \approx 1/N$), and it is necessary to take into account in $I(\alpha)$ only terms linear in α.

Due to the singular character of the integral $I(\alpha)$ [Eq. (29)] one cannot just expand the integrand in power of α. If E belongs to the support of the measure, $-W<E<W$, the change of variable

$$t = \frac{1}{E-e} \tag{31}$$

reduces the integral for $I(\alpha)$ [Eq. (29)] to a sum of two integrals,

$$I(\alpha) = \left(\int_{(E+W)^{-1}}^{\infty} + \int_{-\infty}^{(E-W)^{-1}} \right) (1 - e^{i\alpha t}) \frac{dt}{t^2}, \tag{32}$$

which can be transformed as follows:

$$I(\alpha) = \left(\int_{-\infty}^{\infty} - \int_{(E-W)^{-1}}^{(E+W)^{-1}} \right) (1 - e^{i\alpha t}) \frac{dt}{t^2}. \tag{33}$$

The first integral is equal to $\pi|\alpha|$, and in the second integral one can safely use perturbation theory in α. The final result is

$$I(\alpha) = \pi|\alpha| + i\alpha \ln\frac{W-E}{W+E} + \alpha^2 \frac{W}{E^2-W^2} + O(\alpha^3) \tag{34}$$

and

$$g(\alpha) = \frac{\pi}{W}\delta(\alpha) + \frac{1}{E^2-W^2}. \tag{35}$$

For small values of α

$$e^{-i\alpha} \prod_{j=1}^{N} f(r_j\alpha) = \exp\left[-\frac{N}{2W} v \left(\pi|\alpha| + i\frac{\alpha}{v'} \right) \right], \tag{36}$$

where v plays the role of a ''bare'' coupling constant,

$$v = \frac{1}{N} \sum_{j=1}^{N} r_j, \tag{37}$$

and v' is a ''renormalized'' coupling constant

$$\frac{1}{v'} = \frac{2W}{Nv} + \ln\frac{W-E}{W+E}. \tag{38}$$

The necessity of renormalization for such type of equations is well known when a δ-function potential is added to a d-dimensional system with $d\geqslant 2$ [see, e.g., Ref. [8] and Eqs. (98) and (131)] where it is connected with a one-parameter self-adjoint extension of a singular Hamiltonian. Physically the renormalization means that the limit of infinite small size impurity is not uniquely defined, and depends on internal details of the scatterer. All physically measurable quantities (like the cross section) depend only on the renormalized coupling constant v'. The bare coupling constant v is not observable, and can be arranged to produce any v'. When a specific model of small-size scatterer is considered (e.g., a hard disk with a small radius) one obtains a concrete form of the bare (and renormalized) coupling constant. Below we consider the most interesting case when a renormalized coupling constant is assumed to be independent of N (or energy). All other limits can be derived from this one. Note that in our calculations the appearance of such renormalization [i.e., the

fact that the bare coupling constant v and the renormalization factor $\log(W-E)/(W+E)$ appear only in Eq. (38)] is automatic.

Finally, when $-W<E<W$ the density of state is the sum of two terms

$$\rho_{in}(E)=\frac{N}{2W}-\frac{2W}{(W^2-E^2)(\pi^2+1/v'^2)}. \tag{39}$$

As N is assumed to be large, the first term dominates, and the mean density of levels is

$$\overline{\rho}=\frac{N}{2W}, \tag{40}$$

as it should be.

When E is beyond the interval $[-W,W]$ the calculation is simpler, as in this case there is no singularity on the contour of integration and one can simply expand the integrand of $I(\alpha)$ on series of α:

$$I(\alpha)=i\alpha\ln\frac{E-W}{E+W}+\alpha^2\frac{W}{E^2-W^2}+O(\alpha^3). \tag{41}$$

Therefore,

$$\rho_{out}(E)=\frac{|\phi(E)'|}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\phi^2(E)}{2\sigma^2}\right), \tag{42}$$

where

$$\phi(E)=\ln\frac{E+W}{E-W}-\frac{1}{v'},\quad \sigma^2=\frac{4W^2}{(E^2-W^2)N}. \tag{43}$$

When $N\to\infty$, $\sigma\to 0$, and

$$\rho(E)\to\delta(E-E_c), \tag{44}$$

where E_c is a root of equation $\phi(E_c)=0$,

$$E_c=W\coth\frac{1}{2v'}. \tag{45}$$

These results correspond exactly to what one sees from a simple geometrical picture of the roots of Eq. (16). The poles e_j divide the real axis into $N+1$ intervals. Due to the pole behavior each interval contains one of the solutions, E. There is only one eigenvalue outside of the support of the initial measure, and all other $N-1$ eigenvalues are distributed practically uniformly inside the initial interval $[-W,W]$. The second term in Eq. (39) is a smooth bump, which is necessary to insure that

$$\int_{-W}^{W}\rho_{in}(E)dE=N-1,$$

which can easily be checked by noting that $2W/(E^2-W^2)=\partial(1/v')/\partial E$.

In Eq. (17) we have assumed the particular form of the distribution of $d\mu(e)$, but the results will be valid for any form of this measure (provided that it is not changed noticeably in the scale of the mean distance between levels) with the substitutions $N/2W\to\overline{\rho}$, $E+W\to E-E_{min}$, and $W-E\to E_{max}-E$, where $\overline{\rho}$ is the local mean density of unpertubated levels, and E_{min} and E_{max} are minimal and maximal values of levels included in sum (16).

IV. TWO-POINT CORRELATION FUNCTION

Using the previously discussed method one can compute higher correlation functions as well. Here we consider a calculation of the two-point correlation function, $R_2(E_1,E_2)$, defined in the standard way,

$$R_2(E_1,E_2)=\langle\rho(E_1)\rho(E_2)\rangle, \tag{46}$$

where $\langle\cdots\rangle$ denotes the mean value over all random variables.

For clarity we first consider the case where all residues are equal, $r_j=v$. This case appears, e.g., when a δ-function potential is added to a rectangular billiard with periodic boundary conditions [see Eq. (98)]. A more general case with different r_j will be considered shortly in Sec. VII.

When all residues are the same, our defining equation takes the form

$$\sum_{j=1}^{N}\frac{1}{E-e_j}=\frac{1}{v}, \tag{47}$$

and the two-point correlation function can be expressed as follows:

$$R_2(E_1,E_2)=\left\langle\int\frac{d\alpha_1 d\alpha_2}{4\pi^2}\exp\left[i\sum_{j=1}^{N}\left(\frac{\alpha_1}{E_1-e_j}+\frac{\alpha_2}{E_2-e_j}\right)\right]\times\sum_{k_1,k_2=1}^{N}\frac{1}{(E_1-e_{k_1})^2(E_2-e_{k_2})^2}\times e^{-(i/v)(\alpha_1+\alpha_2)}\right\rangle. \tag{48}$$

After simple algebra this expression can be transformed to

$$\begin{aligned}R_2(E_1,E_2)&=\int\frac{d\alpha_1 d\alpha_2}{4\pi^2}[N(f(\alpha_1,\alpha_2))^{N-1}g(\alpha_1,\alpha_2)\\&+N(N-1)(f(\alpha_1,\alpha_2))^{N-2}\psi_1(\alpha_1,\alpha_2)\psi_2(\alpha_1,\alpha_2)]\\&\times\exp\left(-\frac{i}{v}(\alpha_1+\alpha_2)\right),\end{aligned} \tag{49}$$

where

$$f(\alpha_1,\alpha_2)=\int d\mu(e)\exp\left(i\frac{\alpha_1}{E_1-e}+i\frac{\alpha_2}{E_2-e}\right),$$

$$g(\alpha_1,\alpha_2)=\int d\mu(e)\exp\left(i\frac{\alpha_1}{E_1-e}+i\frac{\alpha_2}{E_2-e}\right)\times\frac{1}{(E_1-e)^2(E_2-e)^2},$$

$$\psi_1(\alpha_1,\alpha_2)=\int d\mu(e)\exp\left(i\frac{\alpha_1}{E_1-e}+i\frac{\alpha_2}{E_2-e}\right)\frac{1}{(E_1-e)^2},\tag{50}$$

$$\psi_2(\alpha_1,\alpha_2)=\int d\mu(e)\exp\left(i\frac{\alpha_1}{E_1-e}+i\frac{\alpha_2}{E_2-e}\right)\frac{1}{(E_2-e)^2}.$$

We shall be interested in the distribution of eigenvalues inside the interval $[-W,W]$, and therefore shall assume that both arguments E_1 and E_2 belong to this interval.

Let us denote

$$f(\alpha_1,\alpha_2)=1-\frac{1}{2W}I(\alpha_1,\alpha_2),\tag{51}$$

where

$$I(\alpha_1,\alpha_2)=\int_{-W}^{W}\left[1-\exp\left(i\frac{\alpha_1}{E_1-e}+i\frac{\alpha_2}{E_2-e}\right)\right]de.\tag{52}$$

Other functions are expressed through $I(\alpha_1,\alpha_2)$ as follows:

$$g(\alpha_1,\alpha_2)=-\frac{1}{2W}\frac{\partial^4}{\partial\alpha_1^2\partial\alpha_2^2}I(\alpha_1,\alpha_2),$$

$$\psi_1(\alpha_1,\alpha_2)=\frac{1}{2W}\frac{\partial^2}{\partial\alpha_1^2}I(\alpha_1,\alpha_2),\tag{53}$$

$$\psi_2(\alpha_1,\alpha_2)=\frac{1}{2W}\frac{\partial^2}{\partial\alpha_2^2}I(\alpha_1,\alpha_2).$$

The integral [Eq. (52)] which defines $I(\alpha_1,\alpha_2)$ can be split into three terms:

$$I(\alpha_1,\alpha_2)=\left(\int_{-\infty}^{\infty}-\int_{-\infty}^{-W}-\int_{\infty}^{W}\right)\times\left[1-\exp\left(i\frac{\alpha_1}{E_1-e}+i\frac{\alpha_2}{E_2-e}\right)\right]de.\tag{54}$$

In the first integral [which we denote by $J(\alpha_1,\alpha_2)$] singular points E_1 and E_2 are on the contour of the integration. In the second and third integrals there are no singularities, and they can be computed in perturbation theory on α_1 and α_2. In the later integrals we will see that one needs only terms linear in α, and

$$I(\alpha_1,\alpha_2)=J(\alpha_1,\alpha_2)+i\left(\alpha_1\ln\frac{W-E_1}{W+E_1}+\alpha_2\ln\frac{W-E_2}{W+E_2}\right).\tag{55}$$

It is the calculation of the first term which is difficult. The details of this calculation are given in the Appendix. The final result for $J(\alpha_1,\alpha_2)$ is the

$$J(\alpha_1,\alpha_2)=\pi(\alpha_1+\alpha_2)\mathrm{sgn}(\alpha_2)-\pi\left[i(\alpha_1+\alpha_2)G\left(-\frac{\alpha_2}{\omega},\frac{\alpha_1}{\omega}\right)+[\alpha_2 J_0(\xi)+i\sqrt{-\alpha_1\alpha_2}J_1(\xi)]\exp\left(i\frac{\alpha_1-\alpha_2}{\omega}\right)\right]\times[\mathrm{sgn}(\alpha_1)-\mathrm{sgn}(\alpha_2)],\tag{56}$$

where $\omega=E_1-E_2$, $\xi=(2/\omega)\sqrt{-\alpha_1\alpha_2}$ and

$$G(x,y)=e^{iy}\int_x^{\infty}J_0(2\sqrt{yt})e^{it}dt.\tag{57}$$

The symmetry relations

$$J(\alpha_2,\alpha_1)=J^*(\alpha_1,\alpha_2),$$

$$J(-\alpha_1,-\alpha_2)=J^*(\alpha_1,\alpha_2),\tag{58}$$

$$J(-\alpha_2,-\alpha_1)=J(\alpha_1,\alpha_2)$$

are also useful. We are interested in the situation when the difference of energies $\omega=E_1-E_2$ is of the order of the mean distance between the levels,

$$\omega=\Omega\frac{2W}{N},\tag{59}$$

and the dimensionless frequency Ω is a constant. In this case one can check that the important values of α will also be of the order of $1/N$, which explains why we have restricted the expansion only up to linear terms. Other simplifications come from the fact that in perturbation theory terms [Eq. (55)] one can set $E_1=E_2$, after which they depend only on the sum $\alpha_1+\alpha_2$.

In the limit of large N one obtains (see the Appendix)

$$f^N(\alpha_1,\alpha_2)=\exp\left(-\frac{N}{2W}\tilde{I}(\alpha_1,\alpha_2)\right),\tag{60}$$

$$g(\alpha_1,\alpha_2)=\frac{1}{2W}\frac{1}{\omega^2}\left(\frac{\partial}{\partial\alpha_1}-\frac{\partial}{\alpha_2}\right)\times\left[\exp\left(i\frac{\alpha_1-\alpha_2}{\omega}\right)\Phi(\alpha_1,\alpha_2)\right],\tag{61}$$

$$\psi_1(\alpha_1,\alpha_2)=\frac{1}{2W}\exp\left(i\frac{\alpha_1-\alpha_2}{\omega}\right)\frac{\partial}{\partial\alpha_1}\Phi(\alpha_1,\alpha_2),\tag{62}$$

$$\psi_2(\alpha_1,\alpha_2) = -\frac{1}{2W}\exp\left(i\frac{\alpha_1-\alpha_2}{\omega}\right)\frac{\partial}{\partial\alpha_2}\Phi(\alpha_1,\alpha_2). \tag{63}$$

Here we introduce

$$\widetilde{I}(\alpha_1,\alpha_2) = J(\alpha_1,\alpha_2) + (\alpha_1+\alpha_2)\frac{i}{v'}, \tag{64}$$

where v' is the renormalized coupling constant as in Eq. (38) and

$$\Phi(\alpha_1,\alpha_2) = 2\pi J_0\left(\frac{2}{\omega}\sqrt{-\alpha_1\alpha_2}\right)\Theta(-\alpha_1\alpha_2)\,\mathrm{sgn}(\alpha_1). \tag{65}$$

Therefore,

$$R_2(\omega) = \int \frac{d\alpha_1 d\alpha_2}{(4\pi W)^2}\left\{ N f^{N-1}\frac{2W}{\omega^2}\left(\frac{\partial}{\partial\alpha_1}-\frac{\partial}{\partial\alpha_2}\right) \times[e^{i[(\alpha_1-\alpha_2)/\omega]}\Phi] - N(N-1) \times f^{N-2}e^{2i[(\alpha_1-\alpha_2)/\omega]}\left[\frac{\partial}{\partial\alpha_1}\Phi\right]\left[\frac{\partial}{\partial\alpha_2}\Phi\right]\right\}. \tag{66}$$

It is convenient to integrate the first term by parts:

$$\begin{aligned}\int\ f^{N-1}&\left(\frac{\partial}{\partial\alpha_1}-\frac{\partial}{\partial\alpha_2}\right)[e^{i[(\alpha_1-\alpha_2)/\omega]}\Phi] \\ &= \frac{N-1}{2W}\int f^{N-2}e^{i[(\alpha_1-\alpha_2)/\omega]}\Phi\left(\frac{\partial}{\partial\alpha_1}-\frac{\partial}{\partial\alpha_2}\right)J \\ &= \frac{N-1}{2W}\int f^{N-2}e^{2i[(\alpha_1-\alpha_2)/\omega]}\Phi^2.\end{aligned} \tag{67}$$

Substituting this expression into the previous equation, one obtains

$$R_2(\omega) = \frac{N(N-1)}{(4\pi W)^2}\int d\alpha_1 d\alpha_2\left\{\frac{\Phi^2}{\omega^2} - \left[\frac{\partial}{\partial\alpha_1}\Phi\right] \times\left[\frac{\partial}{\partial\alpha_2}\Phi\right]\right\} f^{N-2}e^{2i[(\alpha_1-\alpha_2)/\omega]}. \tag{68}$$

The second useful form can be derived by the following transformation of the second term:

$$\begin{aligned}e^{\Psi}&\left[\frac{\partial}{\partial\alpha_2}\Phi\right]\left[\frac{\partial}{\partial\alpha_1}\Phi\right] \\ &= \left[\frac{\partial^2}{2\partial\alpha_1\partial\alpha_2} + \frac{i}{\omega}\left(\frac{\partial}{\partial\alpha_1}-\frac{\partial}{\partial\alpha_2}\right) + \frac{1}{\omega^2}\right]\Phi^2 e^{\Psi},\end{aligned} \tag{69}$$

where

$$\Psi = 2i\frac{\alpha_1-\alpha_2}{\omega}. \tag{70}$$

Combining these two expressions, one obtains

$$R_2(\omega) = -\frac{N(N-1)}{(4\pi W)^2}\int d\alpha_1 d\alpha_2 e^{-(N/2W)\widetilde{I}} \times\left[\frac{\partial^2}{2\partial\alpha_1\partial\alpha_2} + \frac{i}{\omega}\left(\frac{\partial}{\partial\alpha_1}-\frac{\partial}{\partial\alpha_2}\right)\right]\Phi^2 e^{\Psi}. \tag{71}$$

It is easy to check that under the scale transformation (assuming $\lambda>0$)

$$\omega\to\lambda\omega, \quad \alpha_i\to\lambda\alpha_i, \tag{72}$$

the pre-factor does not change, and $\widetilde{I}\to\lambda\widetilde{I}$. Therefore, after the transformations

$$\Omega = \frac{N}{2W}\omega \tag{73}$$

and

$$R_2(\omega) = \frac{N(N-1)}{4W^2}r_2(\Omega), \tag{74}$$

plus the corresponding change of α, the dependence of N will disappear, and after the substitution

$$\alpha_i = \Omega\alpha_i \tag{75}$$

the resulting expression for the two-point correlation function takes the form

$$r_2(\Omega) = -\int\frac{d\alpha_1 d\alpha_2}{4\pi^2}e^{-2\pi\Omega\widetilde{J}} \times\left[\frac{\partial^2}{2\partial\alpha_1\partial\alpha_2} + i\left(\frac{\partial}{\partial\alpha_1}-\frac{\partial}{\partial\alpha_2}\right)\right]\Phi^2 e^{2i(\alpha_1-\alpha_2)}, \tag{76}$$

where

$$\widetilde{J}(\alpha_1,\alpha_2) = \overline{J}(\alpha_1,\alpha_2) + i(\alpha_1+\alpha_2)\frac{1}{2\pi v'}, \tag{77}$$

and, from Eq. (A30),

$$\begin{aligned}\overline{J}(\alpha_1,\alpha_2) = &-\frac{1}{2}(\alpha_1+\alpha_2)\,\mathrm{sgn}(\alpha_2) \\ &+\frac{i}{2}\{(\alpha_1+\alpha_2)G(\alpha_1,-\alpha_2) \\ &-[i\alpha_1 J_0(2\sqrt{-\alpha_1\alpha_2}) \\ &+\sqrt{-\alpha_1\alpha_2}J_1(2\sqrt{-\alpha_1\alpha_2})]e^{i(\alpha_1-\alpha_2)}\} \\ &\times[\mathrm{sgn}(\alpha_1)-\mathrm{sgn}(\alpha_2)].\end{aligned} \tag{78}$$

When $\Omega \to 0$ it is convenient to perform the integration by parts,

$$r_2(\Omega) = -\int d\alpha_1 d\alpha_2 e^{-2\pi\Omega\tilde{J}} \times \left[\frac{\Omega^2}{2} \frac{\partial \tilde{J}}{\partial \alpha_1} \frac{\partial \tilde{J}}{\partial \alpha_1} + \frac{3\Omega i}{8\pi^2} \Phi e^{i(\alpha_1 - \alpha_2)} \right] \Phi^2 e^{2i(\alpha_1 - \alpha_2)}, \tag{79}$$

and take into account only the term linear in Ω,

$$r_2(\Omega) \to \Omega A, \tag{80}$$

where

$$A = -\frac{3i}{8\pi^2} \int d\alpha_1 d\alpha_2 \Phi^3 e^{3i(\alpha_1 - \alpha_2)}. \tag{81}$$

As in the region $\alpha_1 \alpha_2 < 0$ $\Phi = 2\pi J_0(2\sqrt{-\alpha_1 \alpha_2})$, after the change of variables

$$\xi = 2\sqrt{-\alpha_1 \alpha_2}, \quad \eta = -\frac{\alpha_1}{\alpha_2}, \tag{82}$$

one obtains

$$A = -\frac{3\pi i}{2} \int_0^\infty \xi J_0^3(\xi) d\xi \int_0^\infty \frac{d\eta}{\eta} e^{(3\xi i/2)(\eta + \eta^{-1})} + \text{c.c.} \tag{83}$$

The integral over η equals $i\pi H_0^{(1)}(3\xi)$ [see Eq. (A8)] and the final expression for A is

$$A = 3\pi^2 \lim_{\epsilon \to 0} \int_0^\infty \xi J_0[(3+\epsilon)\xi] J_0^3(\xi) d\xi. \tag{84}$$

Here we write $(3+\epsilon)$ (where ϵ is proportional to Ω), as this integral is a discontinuous integral and its value when $\epsilon = 0$ is a half of the value for $\epsilon \to 0$. The last value can be computed using the integral (Ref. [15], p. 414)

$$\int_0^\infty \prod_{n=1}^4 J_0(a_n t) t dt = \frac{1}{\pi^2 \sqrt{a_1 a_2 a_3 a_4}} \begin{cases} K(x) & \text{if } x < 1 \\ \dfrac{1}{x} K(x) & \text{if } x > 1, \end{cases} \tag{85}$$

where $K(x)$ is the full elliptic integral of the second kind,

$$x = \frac{\Delta}{\sqrt{a_1 a_2 a_3 a_4}}, \tag{86}$$

and

$$16\Delta^2 = \prod_{n=1}^4 (a_1 + a_2 + a_3 + a_4 - 2a_n). \tag{87}$$

If the left-hand side is negative the above integral is equal to zero.

In our case $\Delta \to 0$ and $K(0) = \pi/2$; therefore,

$$\lim_{\epsilon \to 0} \int_0^\infty \xi J_0((3+\epsilon)\xi) J_0^3(\xi) d\xi = \frac{1}{2\pi\sqrt{3}}. \tag{88}$$

Hence

$$A = \frac{\pi\sqrt{3}}{2} \approx 2.72 \ldots . \tag{89}$$

Note that the slope at the origin is independent on the coupling constant, and differs from the prediction of the Gaussian orthogonal ensembles of random matrices [$r_2(\Omega) \to (\pi^2/6)\Omega$ [2]].

To find the asymptotics of the two-point correlation function when $\Omega \to \infty$, it is convenient to use Eq. (68). After rescaling of this expression one obtains (the constant term comes from the δ-function contribution of derivatives)

$$r_2(\Omega) = 1 + \left\{ \int_0^\infty d\alpha_1 \int_{-\infty}^0 d\alpha_2 [J_0^2(2\sqrt{-\alpha_1 \alpha_2}) + J_1^2(2\sqrt{-\alpha_1 \alpha_2})] \times \exp[-2\pi\Omega\tilde{J} + 2i(\alpha_1 - \alpha_2)] + \text{c.c.} \right\}. \tag{90}$$

When $\Omega \to \infty$ the dominant contribution comes from the region of small α. Taking into account that when $\alpha \to 0$,

$$\tilde{J} \to \frac{1}{2}(\alpha_1 - \alpha_2) + i\frac{\alpha_1 + \alpha_2}{2\pi v'}, \tag{91}$$

one concludes that the corresponding asymptotics of the two-point correlation function is

$$r_2(\Omega) \to 1 + \frac{2}{\Omega^2(\pi^2 + 1/v'^2)}. \tag{92}$$

Note the absence of oscillation on large Ω, typical of standard random matrix ensembles.

To check the above results we compute the statistical distribution of energy levels of a rectangular billiard with a δ-function potential inside (sometimes called the Seba billiard [9]).

For a rectangle of sides a and b, solutions of the Schrödinger equation

$$(e_{\vec{n}} - \Delta)\psi_{\vec{n}}(\vec{x}) = 0 \tag{93}$$

in two dimensions with periodic boundary conditions have the form

$$\psi_{\vec{n}}(\vec{x}) = \frac{1}{\sqrt{ab}} \exp\left(i\frac{2\pi}{a} n x + i\frac{2\pi}{b} m y \right) \tag{94}$$

and

$$e_{\vec{n}}=\left(\frac{2\pi}{a}n\right)^2+\left(\frac{2\pi}{b}m\right)^2 \tag{95}$$

for all (positive and negative) integers n and m.

As $|\psi_{\vec{n}}(\vec{x})|^2=1/ab$ for all levels, Eq. (12), which determines energy levels after the introduction of a δ-function potential [Eq. (11)], takes the form

$$v\sum_{\vec{n}}\frac{1}{E-e_{\vec{n}}}=1, \tag{96}$$

with $v=\lambda/ab$.

Unperturbed eigenvalues have a multiplicity 4 (for nonzero m, n) due to the existence of positive and negative values of m and n. To remove this degeneracy, in the above sum we consider only positive integers, and to attain the same mean density ($\bar{\rho}=ab/4\pi$) we divide all eigenvalues by 4, after which eigenvalues included in the sum are

$$e_{\vec{n}}=\left(\frac{\pi}{a}n\right)^2+\left(\frac{\pi}{b}m\right)^2, \tag{97}$$

and $m,n>0$.

Sum (96) formally diverges and for computation we consider the renormalization

$$\frac{v'}{\bar{\rho}}\left(\sum_{\vec{n}}\frac{1}{E-e_n}-\bar{\rho}\int_{E_{min}}^{E_{max}}de\,\frac{1}{E-e}\right)=1, \tag{98}$$

where E_{min} and E_{max} are minimal and maximal values of energy included in the sum. The subtracted integral (considered the principal value) is equal to $\log(E_{\max}-E)/(E-E_{min})$, and one obtains the same relation between bare and renormalized coupling constants as before [cf. Eq. (38)]:

$$\frac{1}{v'}=\frac{1}{\bar{\rho}v}+\log\frac{E_{\max}-E}{E-E_{min}}. \tag{99}$$

We take $v'=1$, and compute 100 000 energy levels for such a model. In Fig. 1 the cumulative nearest-neighbor distribution of these levels, $N(s)$, is presented. This quantity is equal to the integral over the nearest-neighbor distribution,

$$N(s)=\int_0^\infty p(s')ds', \tag{100}$$

and it is better defined numerically than the usual nearest-neighbor distribution. In the same figure two other curves are presented. The dashed line corresponds to the Wigner surmise for the cumulative nearest-neighbor distribution in the Gaussian orthogonal ensemble (GOE) of random matrices [2]:

$$N_{GOE}(s)=1-e^{-\pi s^2/4}. \tag{101}$$

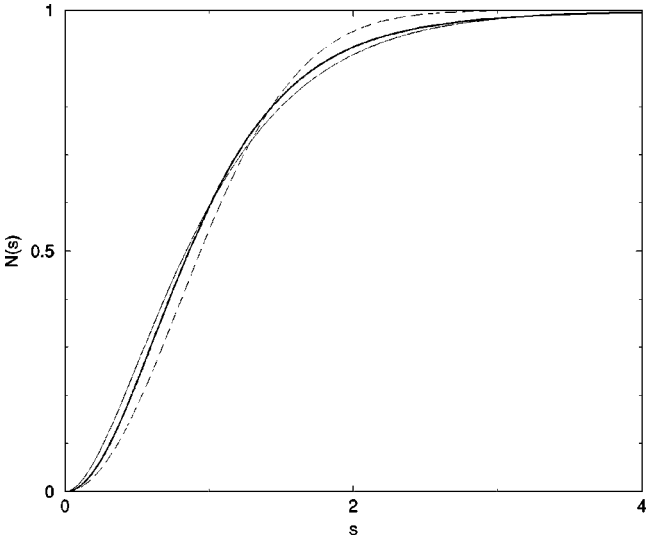

FIG. 1. Nearest-neighbor distribution for a Seba billiard with periodic boundary conditions. The dashed line is the GOE result. The thin line is the semi-Poisson curve.

The thin solid line represents the cumulative nearest-neighbor distribution for the so-called semi-Poisson model [12,13], which serves as a reference point in models with intermediate statistics

$$N_{s.P}(s)=1-(2s+1)e^{-2s}. \tag{102}$$

It is clearly seen that the cumulative nearest-neighbor distribution for the Seba billiard is quite far from the GOE result and it is in between the semi-Poisson curve and the GOE curve.

The numerically computed two-point correlation function for this model is plotted in Fig. 2. The two curves in this

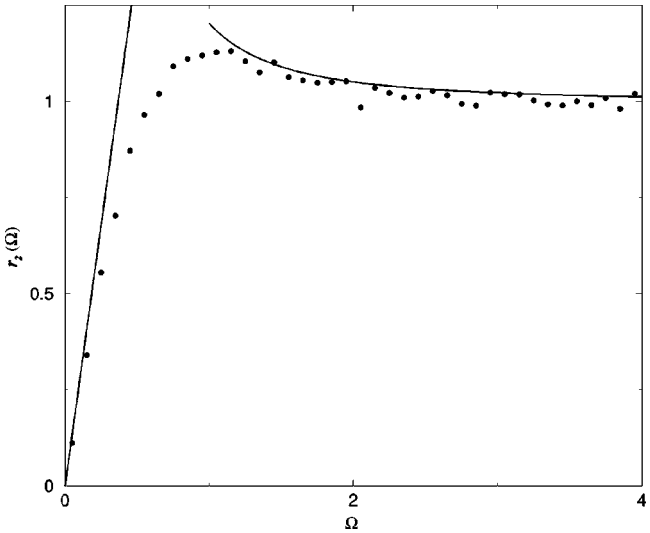

FIG. 2. The two-point correlation function for a Seba billiard with periodic boundary conditions. Solid lines correspond to the asymptotics [Eqs. (80) and (92)] for small and large values of the energy difference.

figure correspond to theoretical predictions for small and large values of the argument given by Eqs. (80) and (92), respectively.

V. SERIES EXPANSIONS

The above expressions are quite cumbersome. Therefore, it is of interest to represent them as power expansions. We start with a function $G(x,y)$ defined in Eq. (A32). It is convenient to define

$$G(x,y)=ig(t,s), \tag{103}$$

where

$$t=ix, \quad s=iy. \tag{104}$$

Using the standard formula for the Bessel function

$$J_n(x)=\sum_{m=0}^{\infty}\left(\frac{x}{2}\right)^{2m+n}\frac{1}{m!(m+n)!}, \tag{105}$$

one obtains

$$g(t,s)=\sum_{m=0}^{\infty}\frac{(-t)^m}{m!}\sum_{n=m}^{\infty}\frac{(-s)^n}{n!}e^{t+s}$$
$$=1-\sum_{m=0}^{\infty}\frac{(-t)^m}{m!}\sum_{n=0}^{m-1}\frac{(-s)^n}{n!}e^{t+s}. \tag{106}$$

Expanding the exponent leads to

$$g(t,s)=1-\sum_{m,n=0}^{\infty}\frac{(-t)^m(-s)^n}{m!n!}R(m,n), \tag{107}$$

$$R(m,n)=\sum_{l=0}^{n}C_n^l(-1)^l\sum_{k=0}^{m-n+l-1}C_m^k(-1)^k, \tag{108}$$

and C_m^n are the binomial coefficients. Only terms for which the upper limits in these sums are non-negative are included in the summation. But

$$\sum_{k=0}^{L}C_m^k(-1)^k=(-1)^LC_{m-1}^L; \tag{109}$$

therefore,

$$R(m,n)=(-1)^{m+n-1}\sum_{l=0}^{n}C_n^lC_{m-1}^{m-n+l-1}$$
$$=(-1)^{m+n-1}C_{n+m-1}^n. \tag{110}$$

Finally we obtain

$$g(t,s)=1+\sum_{m,n=0}^{\infty}\frac{t^ms^n}{m!n!}C_{n+m-1}^n. \tag{111}$$

Using Eq. (A33), one can show that

$$-\frac{1}{\pi i\omega}J(\alpha_1,\alpha_2)=s+t+2\sum_{n,m\geqslant 1}^{\infty}\frac{t^ms^n}{m!n!}C_{n+m-2}^{n-1}, \tag{112}$$

where, as before, $t=-i\alpha_2/\omega$ and $s=i\alpha_1/\omega$.

The expansion of the pre-exponent factor in Eq. (68) can be simplified by the identity (Ref. [15], p. 32)

$$J_n^2(z)=\sum_{m=0}^{\infty}\frac{(-1)^m(2m+2n)!}{m!(m+2n)![(m+n)!]^2}\left(\frac{z}{2}\right)^{2m+2n}. \tag{113}$$

One obtains

$$J_0^2(\xi)+J_1^2(\xi)=1+\sum_{m=1}^{\infty}\frac{(2m)!}{[m!]^3(m+1)!}(\alpha_1\alpha_2)^m. \tag{114}$$

Changing $\alpha_2\rightarrow-\alpha_2$ we can rewrite Eq. (90) in the form

$$r_2(\Omega)=1+\int_0^{\infty}d\alpha_1d\alpha_2P(\alpha_1,\alpha_2)$$
$$\times\exp\left[-\Omega\left(\pi+\frac{i}{v'}\right)\alpha_1-\Omega\left(\pi-\frac{i}{v'}\right)\alpha_2\right]$$
$$\times\exp[2i(\alpha_1+\alpha_2)+2\pi i\Omega Q(\alpha_1,\alpha_2)]+\text{c.c.}, \tag{115}$$

where

$$P(\alpha_1,\alpha_2)=1+\sum_{m=1}^{\infty}\frac{(2m)!}{[m!]^3(m+1)!}(-\alpha_1\alpha_2)^m \tag{116}$$

and

$$Q(\alpha_1,\alpha_2)=\sum_{n,m\geqslant 1}^{\infty}i^{m+n}\frac{\alpha_2^m\alpha_1^n}{m!n!}C_{n+m-2}^{n-1}. \tag{117}$$

VI. LIMITING BEHAVIOR

The above formulas give exact expressions for the two-point correlation function for the problem considered, but they are quite cumbersome and suitable mostly for numerical calculations. The most interesting information which one can extract from them is the behavior of the two-point correlation function at small and large Ω's. The purpose of this section is to discuss methods which permit one to find these asymptotics without a knowledge of the exact solution.

It is clear that in order to find the behavior of the two-point correlation function in the limit $\omega\rightarrow 0$ it is necessary to consider only the case when three initial levels (which we shall denote e_1, e_2, and e_3) are close to each other, and all other levels are far from this triplet. In other words, only three terms in Eq. (47) are large. In such a case Eq. (47), which should determine the positions of the two nearest levels, can be approximated as follows:

$$\frac{1}{E-e_1}+\frac{1}{E-e_2}+\frac{1}{E-e_3}=0. \tag{118}$$

(Note the absence of the coupling constant.) The solution of this equation is

$$E_{1,2}=\frac{e_1+e_2+e_3}{3}\pm\frac{1}{3}\sqrt{e_1^2+e_2^2+e_3^2-e_1e_2-e_1e_3-e_2e_3}. \tag{119}$$

This expression is translationally invariant; therefore, one can assume that $e_1+e_2+e_3=0$ and the difference between two adjacent levels is

$$\Delta E=\frac{2}{\sqrt{3}}\sqrt{e_1^2+e_2^2-e_1e_2}. \tag{120}$$

After corresponding rescaling the two-point correlation function at the limit $\Omega\to 0$ takes the form

$$r_2(\Omega)=\frac{1}{2}\int\delta\left(\Omega-\frac{2}{\sqrt{3}}\sqrt{e_1^2+e_2^2-e_1e_2}\right)de_1de_2. \tag{121}$$

The factor $\frac{1}{2}$ comes from the restriction $e_1<e_2$. Changing variables $e_1=r\cos\theta$ and $e_2=r\sin\theta$, and performing the integral over r, one obtains

$$r_2(\Omega)=\frac{3}{8}\Omega\int\frac{d\theta}{1-\sin\theta\cos\theta}. \tag{122}$$

The last integral is equal to $4\pi/\sqrt{3}$, and finally, in the limit of small Ω we obtain that

$$r_2(\Omega)=\frac{\pi\sqrt{3}}{2}\Omega, \tag{123}$$

which coincides with result (89), obtained above by a different method.

To compute the behavior of the two-point correlation function at large Ω it is convenient to use a method based on the usual trace formula. Let us define

$$G(z)=\sum_{i=1}^{N}\frac{1}{z-e_i}, \tag{124}$$

where all e_i are independent random variables as before.

We need to calculate the density of levels E_j defined by the equation

$$vG(E_j)=1. \tag{125}$$

Formally this density can be expressed as

$$\rho(E)=-\frac{1}{\pi}\mathrm{Im}\left[G(E)+\frac{\partial}{\partial E}\log[1-vG(E)]\right], \tag{126}$$

where the symbol $\mathrm{Im}[F(E)]$ means the limit

$$\mathrm{Im}[F(E)]=\lim_{\epsilon\to 0}\frac{1}{2i}[F(E+i\epsilon)-F(E-i\epsilon)], \tag{127}$$

taken over positive ϵ.

The derivation of Eq. (126) is simple. The function $1-\lambda G(E)$ has zeros at E_j and poles at e_k; therefore,

$$\frac{\partial}{\partial E}\ln[1-vG(E)]=\sum_j\frac{1}{E-E_j}-\sum_k\frac{1}{E-e_k}. \tag{128}$$

The first term in Eq. (126) cancels the poles from unperturbed levels, and the imaginary part produces a δ-function singularity at the required positions.

Let us denote

$$G_\pm(E)=G(E\pm i\epsilon), \tag{129}$$

where ϵ is positive and $\epsilon\to 0$. In calculating the mean values it is useful to take explicitly into account the mean values of $G_\pm(E)$. Using the relation

$$\frac{1}{x\pm i\epsilon}=P\frac{1}{x}\mp i\pi\delta(x), \tag{130}$$

one finds

$$\langle G_\pm(E)\rangle=\mp\pi i\overline{\rho}+\overline{\rho}\log\left(\frac{W+E}{W-E}\right), \tag{131}$$

where $\overline{\rho}$ is the mean level density of the nonperturbed states. Introducing

$$g_\pm(E)=G_\pm(E)-\langle G_\pm(E)\rangle, \tag{132}$$

one can write

$$1-vG_\pm(E)=[1-v\langle G_\pm(E)\rangle][1-\lambda g_\pm(E)], \tag{133}$$

where

$$\lambda_\pm=\frac{v'}{\overline{\rho}(1\pm i\pi v')}, \tag{134}$$

and v' is a renormalized coupling constant as in Eq. (38):

$$\frac{1}{v'}=\frac{1}{\overline{\rho}v}+\ln\frac{W-E}{W+E}. \tag{135}$$

The density of states [ignoring the small correction to the mean density of state as in Eq. (39)] now will take the form

$$\rho(E)=\overline{\rho}-\frac{1}{2\pi i}[g_+(E)-g_-(E)]-\frac{1}{2\pi i}\frac{\partial}{\partial E}\{\log[1-\lambda_+g_+(E)]-\log[1-\lambda_-g_-(E)]\}. \tag{136}$$

The two-point correlation function is the mean value of the product of two such expressions at different energies. The

computation of the mean value can be done in perturbation theory by expanding this expression into powers of $g_{\pm}(E)$ and using a formula

$$\langle g_+^n(E_1) g_-^m(E_2)\rangle \approx \overline{\rho}\int \frac{de}{(E_1-e+i\epsilon)^n (E_2-e-i\epsilon)^m} = 2\pi i(-1)^{n-1}\overline{\rho}\,\overline{C}^{n-1}_{m+n-2}\frac{1}{(\omega)^{m+n-1}}[1+O(\omega)], \quad (137)$$

where $\omega=E_1-E_2$. Therefore, one can organize the perturbation series in a series of inverse power of ω.

Taking into account the first terms in the expansion of the logarithm in the above expression, one obtains

$$\rho(E)=\overline{\rho}-\frac{1}{2\pi i}\left[\left(1-\lambda_+\frac{\partial}{\partial E}\right)g_+(E)-\left(1-\lambda_-\frac{\partial}{\partial E}\right)g_-(E)\right]. \quad (138)$$

At large ω

$$R_2(\omega)=\overline{\rho}^2-\frac{1}{4\pi^2}\left(\lambda_+\frac{\partial}{\partial E_1}g_+(E_1)g_-(E_2) + (E_1\leftrightarrow E_2)+\text{c.c.}\right) = \overline{\rho}^2+\frac{2}{\omega^2(\pi^2+1/v'^2)}, \quad (139)$$

which agrees with Eq. (92) derived from the general formula.

We stress that the methods used in this section are not restricted to particular cases considered. They also can be used in more general situations where exact solutions are not available, e.g., for rank-2 perturbations (two short-range impurities) and similar problems.

VII. GENERAL CASE

In previous sections we considered the calculation of the two-point correlation function under the assumption of equality of all residues. Here a generalization of these calculations to the case of different residues is presented.

When the residues are different, instead of Eq. (47) one has the equation

$$\sum_{j=1}^{N}\frac{r_j}{E-e_j}=1, \quad (140)$$

and, consequently, instead of Eq. (49) one obtains a more general relation

$$R_2(E_1,E_2)=\int_{-\infty}^{\infty}\frac{d\alpha_1 d\alpha_2}{4\pi^2}e^{-i(\alpha_1+\alpha_2)} \times\sum_{k=1}^{N} r_k^2 g(r_k\alpha_1,r_k\alpha_2)\prod_{j\neq k} f(r_j\alpha_1,r_j\alpha_2) + \sum_{k_1\neq k_2} r_{k_1}r_{k_2}\psi_1(r_{k_1}\alpha_1,r_{k_1}\alpha_2) \times\psi_2(r_{k_2}\alpha_1,r_{k_2}\alpha_2)\prod_{j\neq k_1,k_2} f(r_j\alpha_1,r_j\alpha_2), \quad (141)$$

where $f(\alpha_1,\alpha_2)$, $g(\alpha_1,\alpha_2)$, and $\psi_i(\alpha_1,\alpha_2)$ are the same as in Eqs. (50). Repeating the same steps as in Sec. IV, one obtains exact expressions for the two-point correlation function. The analog of Eq. (76) (which is convenient for calculation of the small-Ω series of the two-point correlation function) has the following form:

$$r_2(\Omega)=-\int\frac{d\alpha_1 d\alpha_2}{4\pi^2}e^{-2\pi\Omega\langle\overline{J}(r\alpha_1,r\alpha_2)\rangle-i\Omega v(\alpha_1+\alpha_2)/v'} \times\left[\frac{\partial^2}{2\partial\alpha_1\partial\alpha_2}\langle\Phi(r\alpha_1,r\alpha_2)e^{ir(\alpha_1-\alpha_2)}\rangle^2 + i\left(\frac{\partial}{\partial\alpha_1}-\frac{\partial}{\partial\alpha_2}\right)\langle\Phi(r\alpha_1,r\alpha_2)e^{ir(\alpha_1-\alpha_2)}\rangle \times\langle\Phi(r\alpha_1,r\alpha_2)re^{ir(\alpha_1-\alpha_2)}\rangle\right]. \quad (142)$$

Instead of Eq. (90), useful for large-Ω asymptotics, one obtains

$$r_2(\Omega)=1+\left\{\int_0^{\infty}d\alpha_1\int_{-\infty}^{0}d\alpha_2 e^{-2\pi\Omega\langle\overline{J}(r\alpha_1,r\alpha_2)\rangle-i\Omega v(\alpha_1+\alpha_2)/v'} \times[\langle rJ_0^2(2r\sqrt{-\alpha_1\alpha_2})e^{ir(\alpha_1-\alpha_2)}\rangle^2 + \langle rJ_1^2(2r\sqrt{-\alpha_1\alpha_2})e^{ir(\alpha_1-\alpha_2)}\rangle^2]+\text{c.c.}\right\}. \quad (143)$$

Here $\langle f(r)\rangle$ denotes the mean value over all residues:

$$\langle f(r)\rangle=\frac{1}{N}\sum_{j=1}^{N}f(r_j); \quad (144)$$

functions $\overline{J}(\alpha_1,\alpha_2)$ and $\Phi(\alpha_1,\alpha_2)$ are defined in Eqs. (78) and (A11) and v and v' are ''bare'' and ''renormalized'' coupling constants [see Eqs. (37) and (38)].

As in Sec. IV it is of interest to compute the behavior of the two-point correlation function at small and large energy differences. When $\Omega\to 0$, integration by parts as in Sec. IV leads to

$$r_2(\Omega)\to\Omega A, \tag{145}$$

where

$$A=\int\frac{d\alpha_1 d\alpha_2}{4\pi^2}\left[\frac{i}{2}\langle\Phi(r\alpha_1,r\alpha_2)r^2e^{ir(\alpha_1-\alpha_2)}\rangle \times\langle\Phi(r\alpha_1,r\alpha_2)e^{ir(\alpha_1-\alpha_2)}\rangle^2 +\langle\Phi(r\alpha_1,r\alpha_2)re^{ir(\alpha_1-\alpha_2)}\rangle^2\langle\Phi(r\alpha_1,r\alpha_2)e^{ir(\alpha_1-\alpha_2)}\rangle\right]. \tag{146}$$

Using Eq. (A11) this triple sum is transformed to the form [cf. with Eq. (83)]

$$A=-i\frac{\pi}{N^3}\sum_{i,j,k=1}^{N}\left(\frac{1}{2}r_j^2+r_jr_k\right) \times\int_0^\infty\xi d\xi J_0(r_i\xi)J_0(r_j\xi)J_0(r_k\xi) \times\int_0^\infty\frac{d\eta}{\eta}e^{i(r_i+r_j+r_k)(\eta+1/\eta)\xi/2}+\text{c.c.} \tag{147}$$

According to Eq. (A8) the last integral equals $i\pi H_0^{(1)}[(r_i+r_j+r_k)\xi]$; therefore

$$A=\frac{2\pi^2}{N^3}\sum_{i,j,k=1}^{N}\left(\frac{1}{2}r_j^2+r_jr_k\right) \times\int_0^\infty\xi d\xi J_0(r_i\xi)J_0(r_j\xi)J_0(r_k\xi)J_0[(r_i+r_j+r_k)\xi]. \tag{148}$$

Taking into account Eq. (85), and symmetrizing the answer, one obtains

$$A=\frac{\pi}{6}\frac{1}{N^3}\sum_{i,j,k=1}^{N}\sqrt{\frac{(r_i+r_j+r_k)^3}{r_ir_jr_k}}. \tag{149}$$

Of course, when all residues are equal, $A=\pi\sqrt{3}/2$ as in Eq. (89).

When $\Omega\to\infty$, from Eq. (143) one obtains

$$r_2(\Omega)\to 1+\langle r\rangle^2\int_0^\infty d\alpha_1\int_{-\infty}^0 d\alpha_2 \times e^{-\Omega\langle r\rangle[\pi(|\alpha_1|+|\alpha_2|)]-(i/v')(\alpha_1+\alpha_2)}+\text{c.c.} =1+\frac{2}{\Omega^2(\pi^2+1/v'^2)}, \tag{150}$$

which differs from Eq. (92) only by a suitable definition of the coupling constant.

Note that Eq. (149) is valid only for nonzero values of the residues. Otherwise, the prefactor A formally diverges. This divergence is a consequence of the simple fact that when $r_j=0$ there exist certain energy levels exactly equal to unperturbed levels. Therefore, the set of new energy levels consists of two parts. The first includes energy levels which are changed by the perturbation. Their correlation function is given by the formulas above, where only nonzero residues are taken into account. The second part consists of energy levels which are not changed by the perturbation. Their correlation functions are the same as for the Poisson process and, in particular, they do not display level repulsion. As the cross-correlations between (a finite number) of the old and new energy levels disappear when $N\to\infty$, the resulting statistics is a superposition of two independent distributions and, in general, it will not have level repulsion [i.e., $R_2(\epsilon)\neq 0$ when $\epsilon\to 0$].

The above case is realized e.g., when a δ-function potential [Eq. (11)] is added to a rectangular billiard with the Dirichlet boundary conditions and the positions of the singular point $[\vec{x}_0=(x_0,y_0)]$ are commensurable with the corresponding sides of the rectangular (a and b). In this model unperturbed wave functions are determined by two integers n and m,

$$\psi_{\vec{n}}=\frac{2}{\sqrt{ab}}\sin\left(\frac{\pi}{a}nx\right)\sin\left(\frac{\pi}{b}my\right) \tag{151}$$

and the residues are

$$r_{\vec{n}}=\frac{4}{ab}\sin^2\left(\frac{\pi}{a}nx_0\right)\sin^2\left(\frac{\pi}{b}my_0\right). \tag{152}$$

If

$$\frac{x_0}{a}=\frac{p_1}{q_1},\quad \frac{y_0}{b}=\frac{p_2}{q_2} \tag{153}$$

for coprime integers p_i and q_i there exists only a finite number of different residues depending on values n mod q_1 and m mod q_2. In particular, when n is divisible by q_1 or m is divisible by q_2, $r_{\vec{n}}=0$. This means that for all these values of n and m wave functions and energy eigenvalues will not be changed by the perturbation, and the resulting distribution (included all energy levels) will not describe level repulsion.

Another interesting case corresponds to a model when all residues are also independent random variables with a probability $d\mu(r)$. If r_j never take very small values (more precisely, the mean value of $1/\sqrt{r}$ is finite) the only modification is that a mean value over residues, $\langle f(r)\rangle$, should be taken over the given distribution i.e., instead of Eq. (144) one has to use

$$\langle f(r)\rangle=\int f(r)d\mu(r). \tag{154}$$

In particular the value of the prefactor A is

$$A=\frac{\pi}{6}\int d\mu(r_1)d\mu(r_2)d\mu(r_3)\sqrt{\frac{(r_1+r_2+r_3)^3}{r_1r_2r_3}}. \tag{155}$$

However, if the probability of small values of residues is large, certain expansions should be modified. A natural example is, e.g., the Seba billiard with Dirichlet boundary conditions, when ratios of the positions of the singularity to the corresponding sides [as in Eq. (153)] are noncommensurable irrational numbers. In this case r_n, defined in Eq. (152), are equivalent to random variables

$$r_{\vec{\phi}}=\frac{4}{ab}\sin^2\phi_1\sin^2\phi_2, \tag{156}$$

where angles ϕ_1 and ϕ_2 are uniformly distributed between 0 and π.

Now the two-point correlation function at small Ω will differ from Eq. (145). Indeed, a formal calculation of prefactor (155) shows that it diverges at small r, and that its leading behavior is

$$A\to\frac{\pi}{2}\langle r\rangle\left\langle\frac{1}{\sqrt{r}}\right\rangle^2. \tag{157}$$

However, for variable (156)

$$\langle r\rangle=\frac{4}{\pi^2 ab}\left(\int_0^\pi\sin^2\phi\,d\phi\right)^2=\frac{1}{ab} \tag{158}$$

and

$$\left\langle\frac{1}{\sqrt{r}}\right\rangle=\frac{\sqrt{ab}}{2\pi^2}\left(\int_{\phi_0}^\pi\frac{d\phi}{\sin\phi}\right)^2\approx\frac{\sqrt{ab}}{2\pi^2}\ln^2\phi_0, \tag{159}$$

where ϕ_0 is a cutoff of the integration over ϕ. With logarithmic accuracy ϕ_0 is proportional to Ω, $\phi_0\to\Omega/\Omega_0$, and, when $\Omega\to 0$,

$$r_2(\Omega)\to\frac{1}{8\pi^3}\Omega\ln^4(\Omega/\Omega_0). \tag{160}$$

The results of numerical calculations of 100 000 levels of the Seba billiard with Dirichlet boundary conditions [with irrational ratios (153) and $v'=1$] are presented in Figs. 3 and 4. In Fig. 3 the cumulative nearest-neighbor distribution is plotted. The dashed and thin solid lines are the same as in Fig. 1; the dotted line corresponds to the Poisson distribution

$$N_P(s)=1-e^{-s}. \tag{161}$$

Note that the computed distribution is quite far from all standard examples. Though the resulting distribution is closer to the Poisson distribution than the one for a Seba billiard with periodic boundary conditions (see Fig. 1), one can check that this difference will be present for all nonzero values of the coupling constant (and in particular when $v'\to\infty$).

The two-point correlation function is shown in Fig. 4. The limiting behavior for small [see Eq. (160)] and large [see Eq. (150)] values of arguments are also indicated for comparison by thick solid lines. The value of Ω_0 in Eq. (160), $\Omega_0=52.25$, has been obtained by fitting expression (160) to numerical result for small Ω.

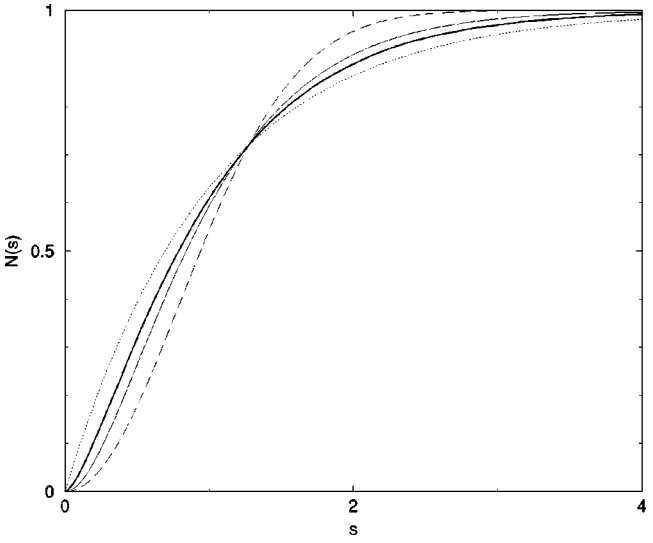

FIG. 3. Nearest-neighbor distribution for a Seba billiard with Dirichlet boundary conditions. The dashed line is the GOE result. The thin line is the semi-Poisson curve. The dotted line is the Poisson prediction.

VIII. CONCLUSION

We have analytically computed the two-point correlation function for zeros of random meromorphic functions with a large number of poles, when these poles are independent random variables. It was demonstrated that the statistics of these zeros corresponds to a distribution with level repulsion which differs from known examples. The resulting distribu-

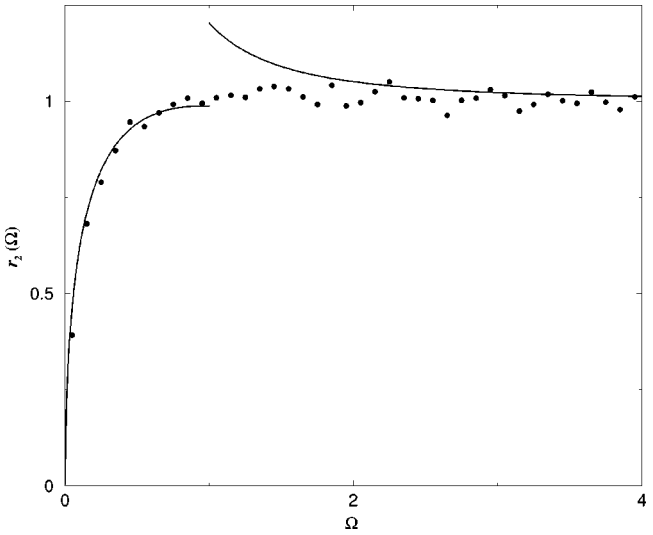

FIG. 4. The two-point correlation function for a Seba billiard with Dirichlet boundary conditions. The solid lines correspond to the asymptotics [Eqs. (160) and (150)] for small and large values of the energy difference.

tion is not universal but depends on residues.

A natural example, where energy levels obey such meromorphic equation, corresponds to a rectangular billiard with a short-range impurity, and our results show the spectral statistics of these models. It is of interest that different boundary conditions give very different results. Even the asymptotic behavior for small energy difference is different [cf. Eqs. (80) and (160)].

We also proposed methods which permit one to find the behavior of the two-point correlation function at small and large arguments without a knowledge of the exact solution. These methods can be applied for cases where exact solutions are not available.

APPENDIX

The purpose of this appendix is to compute the main integral [Eq. (52)]:

$$I(\alpha_1,\alpha_2)=\int_{-W}^{W}\left[1-\exp\left(i\frac{\alpha_1}{E_1-e}+i\frac{\alpha_2}{E_2-e}\right)\right]de. \tag{A1}$$

Let us first derive a few useful relations,

$$\left(\frac{\partial}{\partial\alpha_1}-\frac{\partial}{\partial\alpha_2}\right)J(\alpha_1,\alpha_2) =i\omega\int_{-\infty}^{\infty}\exp\left(i\frac{\alpha_1}{E_1-e}+i\frac{\alpha_2}{E_2-e}\right)\frac{de}{(E_1-e)(E_2-e)}, \tag{A2}$$

where

$$\omega=E_1-E_2. \tag{A3}$$

In this integral we perform a change of variable

$$E_1-e=\frac{\omega}{1+t}. \tag{A4}$$

Now

$$E_2-e=-\frac{t\omega}{1+t},\quad (E_1-e)(E_2-e)=-\frac{t\omega^2}{(1+t)^2},$$

$$de=\frac{\omega}{(1+t)^2}dt. \tag{A5}$$

Hence

$$\left(\frac{\partial}{\partial\alpha_1}-\frac{\partial}{\partial\alpha_2}\right)J(\alpha_1,\alpha_2) =-i\exp\left(i\frac{\alpha_1-\alpha_2}{\omega}\right)\int_{-\infty}^{\infty}\exp\left[i\frac{\alpha_1}{\omega}\left(t-\frac{\alpha_2}{\alpha_1 t}\right)\right]\frac{dt}{t}. \tag{A6}$$

The following standard integrals will be useful for us (see Ref. [14]). When $\mathrm{Im}\,\mu>0$ and $\mathrm{Im}\,\beta^2\mu<0$,

$$\int_0^{\infty}t^{\nu-1}\exp\left[i\frac{\mu}{2}\left(t-\frac{\beta^2}{t}\right)\right]dt=2\beta^{\nu}e^{i\nu\pi/2}K_{-\nu}(\beta\mu), \tag{A7}$$

and when $\mathrm{Im}\,\mu>0$ and $\mathrm{Im}\,\beta^2\mu>0$,

$$\int_0^{\infty}t^{\nu-1}\exp\left[i\frac{\mu}{2}\left(t+\frac{\beta^2}{t}\right)\right]dt=i\pi\beta^{\nu}e^{-i\pi\nu/2}H_{-\nu}^{(1)}(\beta\mu). \tag{A8}$$

Here $K_\nu(x)$ and $H_\nu^{(1)}(x)=J_1(x)+iY_1(x)$ are the Macdonald and Hankel functions, respectively.

Note that $K_0(x)$ is a real function; therefore,

$$\begin{aligned}\int_{-\infty}^{\infty}&\exp\left[i\frac{\alpha_1}{\omega}\left(t-\frac{\alpha_2}{\alpha_1 t}\right)\right]\frac{dt}{t}\\ &=\int_0^{\infty}\exp\left[i\frac{\alpha_1}{\omega}\left(t-\frac{\alpha_2}{\alpha_1 t}\right)\right]\frac{dt}{t}-\text{c.c.}\\ &=\begin{cases}0 & \text{if } \alpha_1\alpha_2>0\\ 2\pi i J_0\left(-\frac{2}{\omega}\sqrt{\alpha_1\alpha_2}\right)\mathrm{sgn}(\alpha_1) & \text{if } \alpha_1\alpha_2<0,\end{cases}\end{aligned} \tag{A9}$$

and $J_0(x)$ is the Bessel function of zero order. Finally,

$$\left(\frac{\partial}{\partial\alpha_1}-\frac{\partial}{\partial\alpha_2}\right)J(\alpha_1,\alpha_2)=\exp\left(i\frac{\alpha_1-\alpha_2}{\omega}\right)\Phi(\alpha_1,\alpha_2), \tag{A10}$$

where

$$\Phi(\alpha_1,\alpha_2)=2\pi J_0\left(\frac{2}{\omega}\sqrt{-\alpha_1\alpha_2}\right)\Theta(-\alpha_1\alpha_2)\mathrm{sgn}(\alpha_1), \tag{A11}$$

$\Theta(x)=1$ if $x>0$, and $\Theta(x)=0$ if $x<0$. Note that

$$\frac{\partial^2}{\partial\alpha_1\partial\alpha_2}\Phi(\alpha_1,\alpha_2)=\frac{1}{\omega^2}\Phi(\alpha_1,\alpha_2). \tag{A12}$$

Using the same method one can prove the following set of equalities:

$$\frac{\partial^2}{\partial\alpha_1\partial\alpha_2}J(\alpha_1,\alpha_2)=-\frac{i}{\omega}\exp\left(i\frac{\alpha_1-\alpha_2}{\omega}\right)\Phi(\alpha_1,\alpha_2). \tag{A13}$$

$$\frac{\partial^2}{\partial\alpha_1^2}J(\alpha_1,\alpha_2)=\exp\left(i\frac{\alpha_1-\alpha_2}{\omega}\right)\frac{\partial}{\partial\alpha_1}\Phi(\alpha_1,\alpha_2). \tag{A14}$$

$$\frac{\partial^2}{\partial\alpha_2^2}J(\alpha_1,\alpha_2)=-\exp\left(i\frac{\alpha_1-\alpha_2}{\omega}\right)\frac{\partial}{\partial\alpha_2}\Phi(\alpha_1,\alpha_2). \tag{A15}$$

$$\frac{\partial^4}{\partial\alpha_1^2\partial\alpha_2^2}J(\alpha_1,\alpha_2)=-\frac{1}{\omega^2}\left(\frac{\partial}{\partial\alpha_1}-\partial\alpha_2\right)\times\left[\exp\left(i\frac{\alpha_1-\alpha_2}{\omega}\right)\Phi(\alpha_1,\alpha_2)\right]. \tag{A16}$$

Note that the differentiation of function Φ produces the δ functions coming from the factor

$$\Theta(-\alpha_1\alpha_2)\mathrm{sgn}(\alpha_1)=\tfrac{1}{2}[\mathrm{sgn}(\alpha_1)-\mathrm{sgn}(\alpha_2)]. \tag{A17}$$

From Eq. (A14) it follows that the second derivative

$$\frac{\partial^2}{\partial\alpha_1^2}J(\alpha_1,\alpha_2)=\phi_1(\alpha_1,\alpha_2) \tag{A18}$$

is known. Therefore,

$$J(\alpha_1,\alpha_2)=J(0,\alpha_2)+\frac{\partial J(0,\alpha_2)}{\partial\alpha_1}+\int_0^{\alpha_1}(\alpha_1-y)\phi_1(y,\alpha_2)dy, \tag{A19}$$

where $J(0,\alpha_2)$ and $\partial J(0,\alpha_2)/\partial\alpha_1$ are values of J and its first derivative at $\alpha_1=0$.

In Sec. III it was demonstrated that

$$J(0,\alpha_2)=\pi|\alpha_2|,$$

and, symmetrically,

$$J(\alpha_1,0)=\pi|\alpha_1|.$$

As the second derivatives is equal to zero when $\alpha_1\alpha_2>0$, the expression of J in these regions, which is continuous when crossing the α_1 and α_2 axes, is

$$J(\alpha_1,\alpha_2)=\pi(\alpha_1+\alpha_2)\mathrm{sgn}(\alpha_2). \tag{A20}$$

It is clear that the function $J(\alpha_1,\alpha_2)$ is a continuous function but with discontinuous first derivatives. The values of these discontinuities follow from the above discontinuity of the function Φ.

Therefore, in the region

$$\alpha_1>0,\ \ \alpha_2<0,\ \ \alpha_1+\alpha_2<0$$

(which is the continuation of the lower left square $\alpha_1<0$, $\alpha_2<0$ through the negative α_2 axis) the function $J(\alpha_1,\alpha_2)$ should take the form

$$J(\alpha_1,\alpha_2)=-\pi(\alpha_2+\alpha_1)+2\pi\alpha_1\exp(-i\alpha_2/\omega)+\int_0^{\alpha_1}(\alpha_1-y)\phi_0(y,\alpha_2)dy, \tag{A21}$$

where

$$\phi_0(\alpha_1,\alpha_2)=\exp\left(i\frac{\alpha_1-\alpha_2}{\omega}\right)\frac{\partial}{\partial\alpha_1}\phi(\alpha_1,\alpha_2), \tag{A22}$$

and

$$\phi(\alpha_1,\alpha_2)=2\pi J_0\left(\frac{2}{\omega}\sqrt{-\alpha_1\alpha_2}\right) \tag{A23}$$

coincides with the function Φ but without a discontinuous factor.

After integration by parts and certain transformations, one obtains that in all regions

$$J(\alpha_1,\alpha_2)=\pi(\alpha_1+\alpha_2)\mathrm{sgn}(\alpha_2)-\pi\left[i(\alpha_1+\alpha_2)G\left(-\frac{\alpha_2}{\omega},\frac{\alpha_1}{\omega}\right)+[\alpha_2J_0(\xi)+i\sqrt{-\alpha_1\alpha_2}J_1(\xi)]\exp\left(i\frac{\alpha_1-\alpha_2}{\omega}\right)\right]\times[\mathrm{sgn}(\alpha_1)-\mathrm{sgn}(\alpha_2)], \tag{A24}$$

where $\xi=(2/\omega)\sqrt{-\alpha_1\alpha_2}$ and

$$G(x,y)=e^{iy}\int_x^\infty J_0(2\sqrt{yt})e^{it}dt. \tag{A25}$$

The function $G(x,y)$ obeys the relations

$$\frac{\partial G(x,y)}{\partial x}=-J_0(\xi)e^{ix+iy}, \tag{A26}$$

$$\frac{\partial G(x,y)}{\partial y}=-i\sqrt{\frac{x}{y}}J_1(\xi)e^{ix+iy}, \tag{A27}$$

where $\xi=2\sqrt{xy}$. To prove the second identity, note that

$$\frac{\partial G(x,y)}{\partial y}=e^{iy}\int_x^\infty\left(iJ_0+J_0'\sqrt{\frac{t}{y}}\right)e^{it}dt. \tag{A28}$$

But the integrand is equal to

$$-i\frac{\partial}{\partial t}[\sqrt{t/y}J_0'(2\sqrt{ty})e^{it})], \tag{A29}$$

and as $J_0'=-J_1$ one obtains the above relation.

The functional equation

$$G(x,y)+G(y,x)=i+iJ_0(\xi)e^{ix+iy} \tag{A30}$$

can be proved by comparing derivatives of both sides, and noting that the integral (Ref. [14], p. 50)

$$\int_0^\infty J_0(at)e^{-\gamma t^2}tdt=\frac{1}{(2\gamma)}\exp\left(-\frac{a^2}{4\gamma}\right) \tag{A31}$$

requires that $G(0,y)=i$.

When $y<x$ the above integral can be taken indefinitely by parts, which leads to the expansion

$$G(x,y)=ie^{ix+iy}\sum_{n=0}^{\infty}(-i\eta)^n J_n(\xi), \tag{A32}$$

where $\eta=\sqrt{y/x}$ and $\xi=2\sqrt{xy}$. The above collection of formulas permits one to find the expansion of $G(x,y)$ for all values of its arguments. Another useful representation of our integral is

$$J(\alpha_1,\alpha_2)=\pi(\alpha_1+\alpha_2)\mathrm{sgn}(\alpha_2)-\pi\omega\Big[i(y-x)G(x,y) + \left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)G(x,y)\Big][\mathrm{sgn}(x)+\mathrm{sgn}(y)], \tag{A33}$$

where $x=-\alpha_2/\omega$ and $y=\alpha_1/\omega$.

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